The Massive Dirac Equation in Kerr Geometry: Separability in Eddington-Finkelstein-type Coordinates and Asymptotics

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ABSTRACT. The separability of the massive Dirac equation in the non-extreme Kerr geometry in horizon-penetrating advanced Eddington-Finkelstein-type coordinates is shown. To this end, Kerr geometry is described by a Carter tetrad and the Dirac spinors and matrices are given in a chiral Newman-Penrose dyad representation. Applying Chandrasekhar's mode ansatz, the Dirac equation is separated into systems of radial and angular ordinary differential equations. Asymptotic radial solutions at infinity, the event horizon, and the Cauchy horizon are explicitly derived. Their decay is analyzed by means of error estimates. Moreover, the eigenfunctions and eigenvalues of the angular system are discussed. Finally, as an application, the scattering of Dirac waves by the gravitational field of a Kerr black hole is studied. This work provides the basis for a Hamiltonian formulation of the massive Dirac equation in Kerr geometry in horizon-penetrating coordinates and for the construction of a functional analytic integral representation of the Dirac propagator.

Contents

I. INTRODUCTION

Over the last five decades, the dynamics of relativistic spin-$\frac{1}{2}$ fermions (Dirac waves) in Kerr geometry was studied extensively by employing different approaches. The probably most established approach is Chandrasekhar's mode analysis [6–8], where the massive Dirac equation is separated by means of time and azimuthal angle modes and

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rewritten in terms of radial wave equations and coupled angular ordinary differential equations (ODEs). Within this framework, many physical processes like the emission and absorption of Dirac waves by rotating black holes or black hole stability under fermionic field perturbations were investigated [3, 27, 29–31]. The dynamics of Dirac waves was also analyzed in the framework of scattering theory [1, 5, 20, 22]. More recently, a functional analytic integral representation of the Dirac propagator was derived using the Hamiltonian formulation of the massive Dirac equation [15–17].

The basis of the mode analysis approach is Chandrasekhar's famous discovery that the massive Dirac equation in Kerr geometry expressed in Boyer-Lindquist coordinates is separable, which was worked out in his original article from 1976 [6] and led to a major breakthrough in the field. At that time, this remarkable result came a bit as a surprise because the Dirac system of coupled first-order partial differential equations (PDEs) was not expected to be separable in Kerr geometry. Despite the tremendous impact of this discovery, the validity of the solutions is naturally restricted to those regions of Kerr geometry where the Boyer-Lindquist coordinates are well-defined. As they have singularities at the event and the Cauchy horizon, respectively, the dynamics of Dirac waves near and across these horizons cannot be described properly. In this article, we resolve this problem by applying a specific analytic extension of the Boyer-Lindquist coordinates that is regular at the horizons, thus, it covers both the exterior and interior black hole regions and allows for well-defined transitions of Dirac waves across the horizons. These so-called advanced Eddington-Finkelstein-type coordinates also feature a coordinate time required for the Hamiltonian formulation of the Dirac equation and the corresponding Cauchy problem. We point out that since the transformation to these coordinates is singular at the horizons and, hence, non-trivial, a careful analysis is necessary. Furthermore, as the mixing of the time and the azimuthal angle variable arising in this transformation leads to a symmetry breaking of structures inherent to Boyer-Lindquist coordinates, it is a priori not clear that the separation of variables property is conserved in advanced Eddington-Finkelstein-type coordinates. In the recent article [11], a similar coordinate system is employed. This work concentrates on the physical aspects of the massive Dirac equation in horizon-penetrating coordinates. More precisely, using numerical methods, the energy spectrum and decay rates of bound states are computed. Here, on the other hand, the focus is on the mathematical aspects, i.e., in the framework of the Newman-Penrose formalism, a detailed analysis of the asymptotics of Dirac waves at infinity, the event horizon, and the Cauchy horizon is given. This includes error estimates and, thus, the proper study of their decay. (For a different horizon-penetrating coordinate system see, e.g., [12].)

In more detail, we perform the mode analysis of massive Dirac waves in the non-extreme Kerr geometry in horizon-penetrating coordinates as follows. First, in Section II, we describe Kerr geometry in the Newman-Penrose formalism by a regular Carter tetrad in advanced Eddington-Finkelstein-type coordinates and calculate the associated spin coefficients. Secondly, in Section III A, we formulate the massive Dirac equation in the chiral representation with a Newman-Penrose dyad basis for the spinor space. General overviews of the Newman-Penrose formalism and of the general relativistic Dirac equation are given in Appendices A and B. Note that the Newman-Penrose formalism is well suited for the analysis of radiative transport in curved spacetimes, especially Dirac wave propagation in Kerr geometry because it can be chosen to reflect symmetries – or be adapted to certain aspects – of the underlying spacetime, which subsequently leads to a reduction in the number of conditional equations and to simplified expressions for geometric quantities. Next, considering a factorization of the Dirac waves in time and azimuthal angle modes, we show the separability of the massive Dirac equation in advanced Eddington-Finkelstein-type coordinates into systems of radial and angular ODEs. Asymptotic radial solutions at infinity, the event horizon, and the Cauchy horizon are determined in Sections III B-III D. Moreover, error estimates that demonstrate suitable decay of these solutions are given. The angular system yields the usual Chandrasekhar-Page equation. The corresponding set of smooth eigenfunctions and the discrete, non-degenerate spectrum of eigenvalues are briefly, however appropriately, discussed in Section III E. Finally, in Section IV, we apply the radial asymptotics at infinity and at the event horizon to the physical problem of scattering of Dirac waves by the gravitational field of a Kerr black hole. To this end, we evaluate the net current of Dirac waves at space-like infinity and at the event horizon. The resulting conservation law for the reflexion and transmission coefficients obtained for horizon-penetrating coordinates is in agreement with that for Boyer-Lindquist coordinates. Therefore, on the one hand, the conserved net current stays positive across the event horizon and, on the other hand, superradiance does not occur. This article provides the basis for a Hamiltonian formulation of the massive Dirac equation in the non-extreme Kerr geometry in horizon-penetrating coordinates. Within this framework, one can construct a functional analytic integral representation of the Dirac propagator that can be applied to Dirac waves with compact support in both the exterior and interior black hole regions. These problems are worked out in [18]. We point out that by using horizon-penetrating coordinates in the construction of the propagator, gluing techniques to connect the exterior and interior regions of the black hole, as employed in the Boyer-Lindquist case, are no longer required.

II. NEWMAN-PENROSE REPRESENTATION OF KERR GEOMETRY IN HORIZON-PENETRATING ADVANCED EDDINGTON-FINKELSTEIN-TYPE COORDINATES

The non-extreme Kerr geometry [23] is a connected, orientable and time-orientable, smooth, asymptotically flat Lorentzian 4-manifold $(\mathfrak{M}, \boldsymbol{g})$ with topology $S^2 \times \mathbb{R}^2$ that consists of a differentiable manifold $\mathfrak{M}$ and a stationary, axisymmetric Lorentzian metric $\boldsymbol{g}$ with signature $(1,3)$, which in Boyer-Lindquist coordinates (t, r, θ, φ) [2], where $t \in \mathbb{R}, r \in \mathbb{R}_{>0}, \theta \in [0, \pi]$, and $\varphi \in [0, 2\pi)$, is given by

$$\begin{aligned}\boldsymbol{g} = {} & \frac{\Delta}{\Sigma} \big(\mathrm{d}t - a \sin^2(\theta)\, \mathrm{d}\varphi\big) \otimes \big(\mathrm{d}t - a \sin^2(\theta)\, \mathrm{d}\varphi\big) \\ & - \frac{\sin^2(\theta)}{\Sigma} \Big(\big[r^2 + a^2\big] \mathrm{d}\varphi - a\, \mathrm{d}t\Big) \otimes \Big(\big[r^2 + a^2\big] \mathrm{d}\varphi - a\, \mathrm{d}t\Big) - \frac{\Sigma}{\Delta} \mathrm{d}r \otimes \mathrm{d}r - \Sigma\, \mathrm{d}\theta \otimes \mathrm{d}\theta\,.\end{aligned} \tag{1}$$

The horizon function is defined by $\Delta = \Delta(r) := (r - r_+)(r - r_-) = r^2 - 2Mr + a^2$, $r_\pm := M \pm \sqrt{M^2 - a^2}$ denote the event and the Cauchy horizon, respectively, M is the mass and aM the angular momentum of the black hole with $0 \leq a < M$, and $\Sigma = \Sigma(r, \theta) := r^2 + a^2 \cos^2(\theta)$. We describe Kerr geometry in terms of a local Newman-Penrose null tetrad frame that is adapted to the principal null geodesics, i.e., the tetrad coincides with the two principal null directions of the Weyl tensor. In this so-called Kinnersley frame [24], since Kerr geometry is algebraically special and of Petrov type D, we are presented with the computational advantage that the four spin coefficients κ, σ, λ, and ν vanish and only one Weyl scalar, namely Ψ_2, is non-zero. Accordingly, the congruences formed by the two principal null directions must be geodesic and shear-free [26]. We construct the Kinnersley tetrad directly from the tangent vectors of the principal null geodesics [8]

$$\frac{\mathrm{d}t}{\mathrm{d}\chi} = \frac{r^2 + a^2}{\Delta} E\,, \quad \frac{\mathrm{d}r}{\mathrm{d}\chi} = \pm E\,, \quad \frac{\mathrm{d}\theta}{\mathrm{d}\chi} = 0\,, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}\chi} = \frac{a}{\Delta} E\,, \tag{2}$$

where χ is an affine parameter and E denotes a constant. Hence, aligning the real-valued Newman-Penrose vectors $\boldsymbol{l}$ and $\boldsymbol{n}$ with the principal null directions and choosing conjugate-complex Newman-Penrose vectors $\boldsymbol{m}$ and $\overline{\boldsymbol{m}}$ in such a way that they satisfy the Newman-Penrose conditions (A1)-(A3) yields

$$\begin{aligned}\boldsymbol{l} &= \frac{1}{|\Delta|} \Big(\big[r^2 + a^2\big] \partial_t + \Delta\, \partial_r + a\, \partial_\varphi\Big) \\ \boldsymbol{n} &= \frac{\mathrm{sign}(\Delta)}{2\,\Sigma} \Big(\big[r^2 + a^2\big] \partial_t - \Delta\, \partial_r + a\, \partial_\varphi\Big) \\ \boldsymbol{m} &= \frac{1}{\sqrt{2}\,\big(r + \mathrm{i}a \cos(\theta)\big)} \big(\mathrm{i}a \sin(\theta)\, \partial_t + \partial_\theta + \mathrm{i} \csc(\theta)\, \partial_\varphi\big) \\ \overline{\boldsymbol{m}} &= -\frac{1}{\sqrt{2}\,\big(r - \mathrm{i}a \cos(\theta)\big)} \big(\mathrm{i}a \sin(\theta)\, \partial_t - \partial_\theta + \mathrm{i} \csc(\theta)\, \partial_\varphi\big)\end{aligned} \tag{3}$$

with the signum function

$$\mathrm{sign}(\Delta) := \begin{cases} +1 & \text{for } \Delta \geq 0 \\ -1 & \text{for } \Delta < 0\,. \end{cases}$$

Note that the use of the absolute value and the signum function allows for a unified representation of the frame both outside and inside the event and the Cauchy horizon, respectively. For the calculation of the corresponding spin coefficients, i.e., for solving the first Maurer-Cartan equation of structure (A5), we require the dual co-tetrad of (3) given by

$$\begin{aligned}\boldsymbol{l} &= \mathrm{sign}(\Delta) \bigg(\mathrm{d}t - \frac{\Sigma}{\Delta} \mathrm{d}r - a \sin^2(\theta)\, \mathrm{d}\varphi\bigg) \\ \boldsymbol{n} &= \frac{|\Delta|}{2\,\Sigma} \bigg(\mathrm{d}t + \frac{\Sigma}{\Delta} \mathrm{d}r - a \sin^2(\theta)\, \mathrm{d}\varphi\bigg)\end{aligned}$$

$$\boldsymbol{m} = \frac{1}{\sqrt{2}\left(r + \mathrm{i}a\cos(\theta)\right)}\Big(\mathrm{i}a\sin(\theta)\,\mathrm{d}t - \Sigma\,\mathrm{d}\theta - \mathrm{i}\big[r^2 + a^2\big]\sin(\theta)\,\mathrm{d}\varphi\Big)$$

$$\overline{\boldsymbol{m}} = -\frac{1}{\sqrt{2}\left(r - \mathrm{i}a\cos(\theta)\right)}\Big(\mathrm{i}a\sin(\theta)\,\mathrm{d}t + \Sigma\,\mathrm{d}\theta - \mathrm{i}\big[r^2 + a^2\big]\sin(\theta)\,\mathrm{d}\varphi\Big)\,.$$

We introduce further computational simplifications by applying a class III local Lorentz transformation (A7) with parameters of the form

$$\varsigma = \sqrt{\frac{|\Delta|}{2\,\Sigma}} \quad \text{and} \quad e^{\mathrm{i}\psi} = \frac{\sqrt{\Sigma}}{r - \mathrm{i}a\cos(\theta)}$$

to the Kinnersley tetrad (3). This transformation leads to the so-called Carter (symmetric) frame [4], which acts under the composition of the discrete time and azimuthal angle reversal isometries $t \mapsto -t$ and $\varphi \mapsto -\varphi$ as

$$\boldsymbol{l}' \mapsto -\mathrm{sign}(\Delta)\,\boldsymbol{n}'\,, \quad \boldsymbol{n}' \mapsto -\mathrm{sign}(\Delta)\,\boldsymbol{l}'\,, \quad \boldsymbol{m}' \mapsto \overline{\boldsymbol{m}}'\,, \quad \overline{\boldsymbol{m}}' \mapsto \boldsymbol{m}'$$

and, thus, has only six independent spin coefficients

$$\kappa' = -\nu'\,, \quad \pi' = -\tau'\,, \quad \alpha' = -\beta'\,, \quad \sigma' = \mathrm{sign}(\Delta)\,\lambda'\,, \quad \mu' = \mathrm{sign}(\Delta)\,\varrho'\,, \quad \epsilon' = \mathrm{sign}(\Delta)\,\gamma'\,.$$

We obtain

$$\begin{aligned}
\boldsymbol{l}' &= \frac{1}{\sqrt{2\,\Sigma\,|\Delta|}}\Big(\big[r^2 + a^2\big]\partial_t + \Delta\,\partial_r + a\,\partial_\varphi\Big)\\
\boldsymbol{n}' &= \frac{\mathrm{sign}(\Delta)}{\sqrt{2\,\Sigma\,|\Delta|}}\Big(\big[r^2 + a^2\big]\partial_t - \Delta\,\partial_r + a\,\partial_\varphi\Big)\\
\boldsymbol{m}' &= \frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_t + \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\varphi\big)\\
\overline{\boldsymbol{m}}' &= -\frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_t - \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\varphi\big)\,.
\end{aligned} \tag{4}$$

The dual co-tetrad reads

$$\begin{aligned}
\boldsymbol{l}' &= \sqrt{\frac{|\Delta|}{2\,\Sigma}}\,\mathrm{sign}(\Delta)\left(\mathrm{d}t - \frac{\Sigma}{\Delta}\,\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\varphi\right)\\
\boldsymbol{n}' &= \sqrt{\frac{|\Delta|}{2\,\Sigma}}\left(\mathrm{d}t + \frac{\Sigma}{\Delta}\,\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\varphi\right)\\
\boldsymbol{m}' &= \frac{1}{\sqrt{2\,\Sigma}}\Big(\mathrm{i}a\sin(\theta)\,\mathrm{d}t - \Sigma\,\mathrm{d}\theta - \mathrm{i}\big[r^2 + a^2\big]\sin(\theta)\,\mathrm{d}\varphi\Big)\\
\overline{\boldsymbol{m}}' &= -\frac{1}{\sqrt{2\,\Sigma}}\Big(\mathrm{i}a\sin(\theta)\,\mathrm{d}t + \Sigma\,\mathrm{d}\theta - \mathrm{i}\big[r^2 + a^2\big]\sin(\theta)\,\mathrm{d}\varphi\Big)\,.
\end{aligned} \tag{5}$$

We point out that Boyer-Lindquist coordinates have singularities at the event and the Cauchy horizon. The light cone of an observer that, for example, approaches the event horizon from outside the black hole closes up and becomes degenerate (see Figure 1). Moreover, space and time reverse their roles in between the event and the Cauchy horizon. All this precludes a study of the propagation of Dirac waves across the horizons. In order to have a consistent description of their overall dynamics, we use advanced Eddington-Finkelstein-type coordinates (see [13, 14] for the original Eddington-Finkelstein null coordinates). This analytic extension covers the black

hole exterior and interior and allows for regular transitions of Dirac waves across the horizons. Furthermore, it possesses a proper coordinate time, which is required for the Hamiltonian formulation of the Dirac equation and the corresponding Cauchy problem. The advanced Eddington-Finkelstein-type coordinates are constructed as follows. By means of the tangent vectors (2), we derive two relations between the time and the radial coordinate and two relations between the azimuthal angle and the radial coordinate along the principal null geodesics

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm\frac{r^2+a^2}{\Delta} \quad \Rightarrow \quad t = \pm\int \frac{r^2+a^2}{\Delta}\,\mathrm{d}r + c_\pm = \pm r_\star + c_\pm \tag{6}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \pm\frac{a}{\Delta} \quad \Rightarrow \quad \varphi = \pm\int \frac{a}{\Delta}\,\mathrm{d}r + c'_\pm = \pm\frac{a}{r_+ - r_-}\,\ln\left|\frac{r-r_+}{r-r_-}\right| + c'_\pm\,,$$

where

$$r_\star := r + \frac{r_+^2+a^2}{r_+-r_-}\,\ln|r-r_+| - \frac{r_-^2+a^2}{r_+-r_-}\,\ln|r-r_-|$$

is the Regge-Wheeler coordinate and $c_\pm, c'_\pm$ are constants of integration, motivating a transformation to coordinates that are adapted to ingoing null geodesics

$$\mathbb{R}\times\mathbb{R}_{>0}\times[0,\pi]\times[0,2\pi)\to\mathbb{R}\times\mathbb{R}_{>0}\times[0,\pi]\times[0,2\pi)\,,\qquad (t,r,\theta,\varphi)\mapsto(\tau,r,\theta,\phi)$$

with

$$\begin{aligned}\tau &:= t + r_\star - r = t + \frac{r_+^2+a^2}{r_+-r_-}\,\ln|r-r_+| - \frac{r_-^2+a^2}{r_+-r_-}\,\ln|r-r_-| \\ \phi &:= \varphi + \frac{a}{r_+-r_-}\,\ln\left|\frac{r-r_+}{r-r_-}\right|.\end{aligned} \tag{7}$$

In these so-called advanced Eddington-Finkelstein-type coordinates, light cones are non-degenerate at the horizons (see Figure 2). Approaching again the event horizon from outside the black hole, the light cone tips over until its outgoing future light cone is aligned with the horizon, which indicates its trapping characteristic. Besides, ingoing light rays are represented simply by straight lines. This can be directly seen from the two relations between the advanced Eddington-Finkelstein-type time and radial coordinates for ingoing and outgoing null geodesics

$$\frac{\mathrm{d}\tau}{\mathrm{d}r}_{|\mathrm{in}} = -1 \quad \text{and} \quad \frac{\mathrm{d}\tau}{\mathrm{d}r}_{|\mathrm{out}} = 1 + \frac{4Mr}{\Delta}\,,$$

which are obtained by inserting (7) into (6). The metric (1) represented in these coordinates becomes

$$\begin{aligned}\boldsymbol{g} = {}&\left(1-\frac{2Mr}{\Sigma}\right)\mathrm{d}\tau\otimes\mathrm{d}\tau - \frac{2Mr}{\Sigma}\Big(\big[\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\big]\otimes\mathrm{d}\tau + \mathrm{d}\tau\otimes\big[\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\big]\Big) \\ &-\left(1+\frac{2Mr}{\Sigma}\right)\big(\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\big)\otimes\big(\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\big) - \Sigma\,\mathrm{d}\theta\otimes\mathrm{d}\theta - \Sigma\sin^2(\theta)\,\mathrm{d}\phi\otimes\mathrm{d}\phi\,.\end{aligned}$$

Considering the restriction to constant-τ hypersurfaces reveals that these are space-like and that τ is a proper coordinate time. In the Carter-Penrose diagrams shown in Figure 3, we depict the constant-t and constant-r hypersurfaces in Boyer-Lindquist coordinates and the constant-τ and constant-r hypersurfaces in advanced Eddington-Finkelstein-type coordinates for Kerr geometry. While the constant-t hypersurfaces become time-like inside the black hole in region II, the constant-τ hypersurfaces are always space-like and smoothly continued across the horizons. The Carter tetrad (4) and its dual (5) expressed in advanced Eddington-Finkelstein-type coordinates read

$$\boldsymbol{l}' = \frac{1}{\sqrt{2\,\Sigma\,|\Delta|}}\big([\Delta+4Mr]\,\partial_\tau + \Delta\,\partial_r + 2a\,\partial_\phi\big)$$

$$\boldsymbol{n}' = \sqrt{\frac{|\Delta|}{2\,\Sigma}}\big(\partial_\tau - \partial_r\big)$$

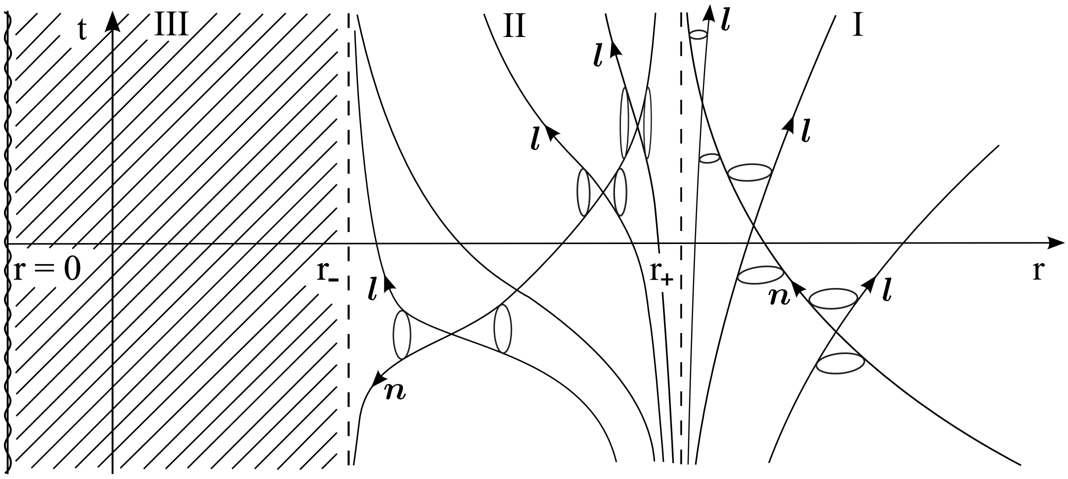

FIG. 1: Causal structure of the non-extreme Kerr geometry in Boyer-Lindquist coordinates. A projection onto the (t, r)-plane, where every point is a 2-sphere, is presented. The real-valued Newman-Penrose null vectors $\boldsymbol{l}$ and $\boldsymbol{n}$, which point along the principal null directions, form the light cones. The light cone of an observer approaching the event horizon from outside the black hole ($r \searrow r_+$) closes up and becomes degenerate. In contrast, it opens up when the observer approaches the event horizon from inside the black hole ($r \nearrow r_+$). This stems from the fact that the roles of space and time are reversed in the black hole interior region II. When $r \to \infty$, the light cone becomes a 45°-Minkowski light cone because the spacetime is asymptotically flat. Note that all figures are restricted to regions I and II in order to avoid the issues that arise when one considers the ring singularity at $(r = 0, \theta = \pi/2)$ and the maximum analytic extension.

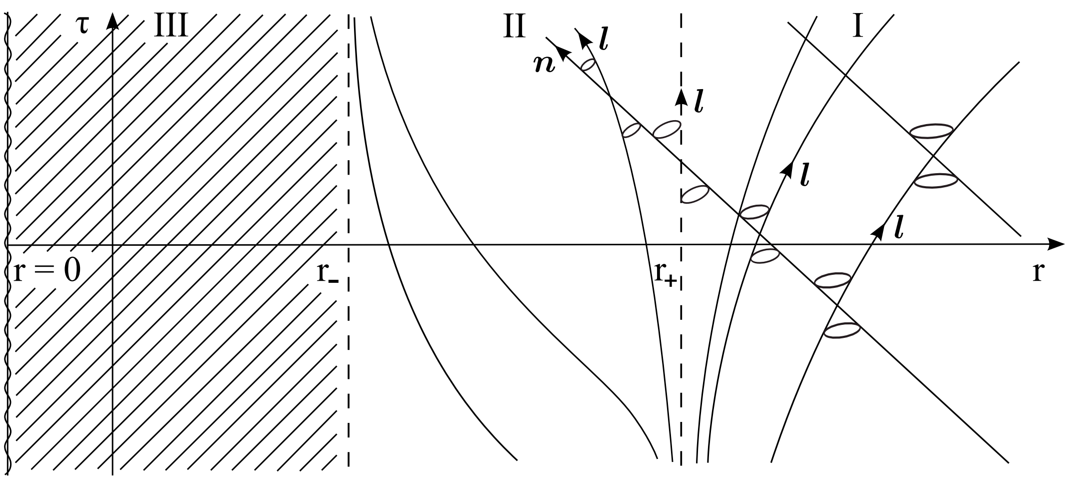

FIG. 2: Causal structure of the non-extreme Kerr geometry in advanced Eddington-Finkelstein-type coordinates. A projection onto the (τ, r)-plane is presented. Ingoing light rays are straight lines pointing in the $\boldsymbol{n}$-direction. The light cone of an observer moving toward the event horizon from outside the black hole tips over until – after having crossed the event horizon – its future light cone is completely in the black hole interior. This shows the trapping characteristic of event horizons.

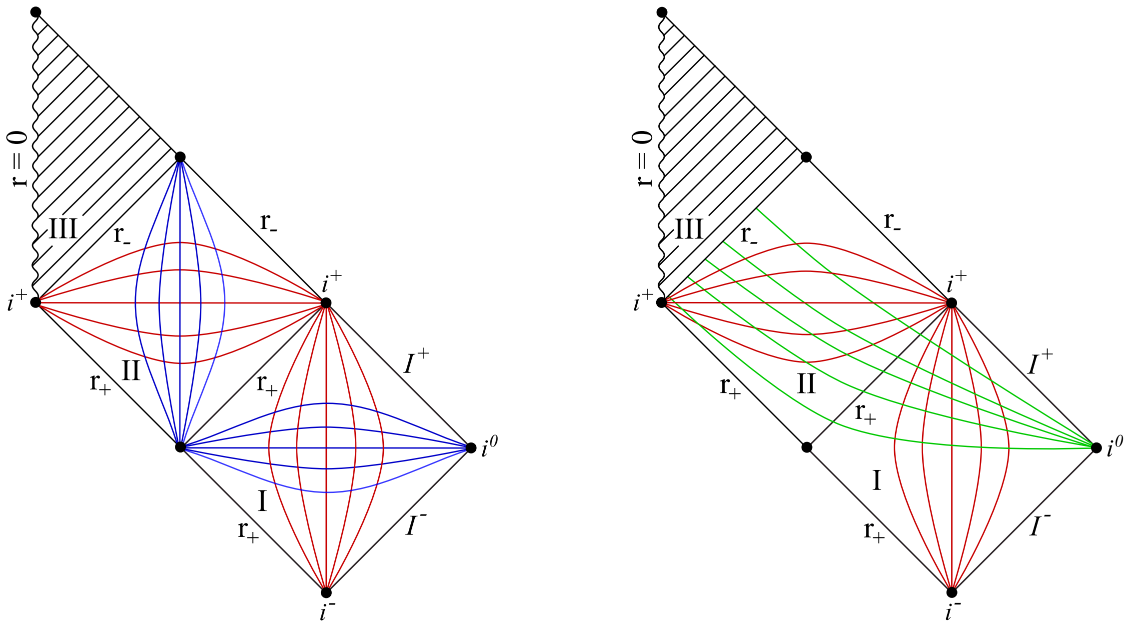

FIG. 3: Carter-Penrose diagrams for the non-extreme Kerr geometry in Boyer-Lindquist coordinates (left) and advanced Eddington-Finkelstein-type coordinates (right). The blue lines represent constant-t hypersurfaces, the red lines constant-r hypersurfaces, and the green lines constant-τ hypersurfaces. The constant-t and constant-r hypersurfaces are restricted to either the exterior or the interior of the black hole. Their nature changes across the event horizon, i.e., space-like hypersurfaces become time-like and vice versa. The constant-τ hypersurfaces (cut-off at the Cauchy horizon) are space-like outside and inside the black hole and smooth across the event horizon.

$$\boldsymbol{m}' = \frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_\tau + \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\phi\big)$$

$$\overline{\boldsymbol{m}}' = -\frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_\tau - \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\phi\big)$$

and

$$\boldsymbol{l}' = \sqrt{\frac{|\Delta|}{2\,\Sigma}}\,\mathrm{sign}(\Delta)\left(\mathrm{d}\tau + \left[1 - \frac{2\Sigma}{\Delta}\right]\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\right)$$

$$\boldsymbol{n}' = \sqrt{\frac{|\Delta|}{2\,\Sigma}}\big(\mathrm{d}\tau + \mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\big)$$

$$\boldsymbol{m}' = \frac{1}{\sqrt{2\,\Sigma}}\Big(\mathrm{i}a\sin(\theta)\,[\mathrm{d}\tau + \mathrm{d}r] - \Sigma\,\mathrm{d}\theta - \mathrm{i}\big[r^2 + a^2\big]\sin(\theta)\,\mathrm{d}\phi\Big)$$

$$\overline{\boldsymbol{m}}' = -\frac{1}{\sqrt{2\,\Sigma}}\Big(\mathrm{i}a\sin(\theta)\,[\mathrm{d}\tau + \mathrm{d}r] + \Sigma\,\mathrm{d}\theta - \mathrm{i}\big[r^2 + a^2\big]\sin(\theta)\,\mathrm{d}\phi\Big)\,.$$

Since the real-valued Newman-Penrose vector $\boldsymbol{l}'$ is still singular at the horizons, we apply a class III local Lorentz transformation (A7) with parameters

$$\varsigma = \frac{\sqrt{|\Delta|}}{r_+} \quad \text{and} \quad \psi = 0\,.$$

This leads to the regular Carter tetrad

$$
\begin{aligned}
\boldsymbol{l}'' &= \frac{1}{\sqrt{2\,\Sigma}\,r_+}\big([\Delta + 4Mr]\,\partial_\tau + \Delta\,\partial_r + 2a\,\partial_\phi\big) \\
\boldsymbol{n}'' &= \frac{r_+}{\sqrt{2\,\Sigma}}\big(\partial_\tau - \partial_r\big) \\
\boldsymbol{m}'' &= \frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_\tau + \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\phi\big) \\
\overline{\boldsymbol{m}}'' &= -\frac{1}{\sqrt{2\,\Sigma}}\big(\mathrm{i}a\sin(\theta)\,\partial_\tau - \partial_\theta + \mathrm{i}\csc(\theta)\,\partial_\phi\big)
\end{aligned}
\tag{8}
$$

and to the dual co-tetrad

$$
\begin{aligned}
\boldsymbol{l}'' &= \frac{\Delta}{\sqrt{2\,\Sigma}\,r_+}\left(\mathrm{d}\tau + \left[1 - \frac{2\Sigma}{\Delta}\right]\mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\right) \\
\boldsymbol{n}'' &= \frac{r_+}{\sqrt{2\,\Sigma}}\big(\mathrm{d}\tau + \mathrm{d}r - a\sin^2(\theta)\,\mathrm{d}\phi\big) \\
\boldsymbol{m}'' &= \frac{1}{\sqrt{2\,\Sigma}}\Big(\mathrm{i}a\sin(\theta)\,[\mathrm{d}\tau + \mathrm{d}r] - \Sigma\,\mathrm{d}\theta - \mathrm{i}\big[r^2 + a^2\big]\sin(\theta)\,\mathrm{d}\phi\Big) \\
\overline{\boldsymbol{m}}'' &= -\frac{1}{\sqrt{2\,\Sigma}}\Big(\mathrm{i}a\sin(\theta)\,[\mathrm{d}\tau + \mathrm{d}r] + \Sigma\,\mathrm{d}\theta - \mathrm{i}\big[r^2 + a^2\big]\sin(\theta)\,\mathrm{d}\phi\Big)\,.
\end{aligned}
$$

Substituting the latter into the first Maurer-Cartan equation of structure (A5), we obtain regular spin coefficients for the non-extreme Kerr geometry in advanced Eddington-Finkelstein-type coordinates

$$
\begin{aligned}
&\kappa'' = \sigma'' = \lambda'' = \nu'' = 0\,, \qquad \alpha'' = -\beta'' = -\frac{1}{(2\,\Sigma)^{3/2}}\Big(\big[r^2 + a^2\big]\cot(\theta) - \mathrm{i}ra\sin(\theta)\Big)\,, \\
&\pi'' = -\tau'' = \frac{\mathrm{i}a\sin(\theta)}{\sqrt{2\,\Sigma}\,\big(r - \mathrm{i}a\cos(\theta)\big)}\,, \qquad \mu'' = -\frac{r_+}{\sqrt{2\,\Sigma}\,\big(r - \mathrm{i}a\cos(\theta)\big)}\,, \qquad \varrho'' = -\frac{\Delta}{\sqrt{2\,\Sigma}\,r_+\big(r - \mathrm{i}a\cos(\theta)\big)}\,, \\
&\gamma'' = -\frac{r_+}{2^{3/2}\,\sqrt{\Sigma}\,\big(r - \mathrm{i}a\cos(\theta)\big)}\,, \qquad \epsilon'' = \frac{r^2 - a^2 - 2\mathrm{i}a\cos(\theta)\,(r - M)}{2^{3/2}\,\sqrt{\Sigma}\,r_+\big(r - \mathrm{i}a\cos(\theta)\big)}\,.
\end{aligned}
\tag{9}
$$

In the next section, we show the separability of the massive Dirac equation in this horizon-penetrating frame, perform asymptotic analyses of the solution of the resulting radial system at space-like infinity, at the event horizon, and at the Cauchy horizon, and discuss the eigenfunctions as well as the eigenvalues of the resulting angular system.

III. THE MASSIVE DIRAC EQUATION IN THE ANALYTICALLY EXTENDED KERR GEOMETRY

A. Mode Ansatz and Separability

Substituting the regular Carter tetrad (8) as well as the associated spin coefficients (9) into the massive Dirac equation (B5) and employing Chandrasekhar's mode ansatz (see, e.g., [8])

$$
\begin{aligned}
\mathscr{F}_i(\tau, r, \theta, \phi) &= e^{-\mathrm{i}(\omega\tau + k\phi)}\,\big(r - \mathrm{i}a\cos(\theta)\big)^{-1/2}\,\mathscr{H}_i(r, \theta) \\
\mathscr{G}_i(\tau, r, \theta, \phi) &= e^{-\mathrm{i}(\omega\tau + k\phi)}\,\big(r + \mathrm{i}a\cos(\theta)\big)^{-1/2}\,\mathscr{J}_i(r, \theta)\,,
\end{aligned}
\tag{10}
$$

where $\omega \in \mathbb{R}$ is the frequency, $k \in \mathbb{Z}+1/2$ the wave number, and $i \in \{1,2\}$, we obtain the first-order PDE system

$$r_+^{-1}\big(\Delta\,\partial_r + r - M - \mathrm{i}\omega\,(\Delta + 4Mr) - 2\mathrm{i}ak\big)\mathscr{H}_1 + \big(\partial_\theta + 2^{-1}\cot(\theta) - a\omega\sin(\theta) - k\csc(\theta)\big)\mathscr{H}_2$$
$$= \sqrt{2}\mathrm{i}\mu_\star\big(r - \mathrm{i}a\cos(\theta)\big)\,\mathscr{J}_1$$

$$r_+\big(\partial_r + \mathrm{i}\omega\big)\mathscr{H}_2 - \big(\partial_\theta + 2^{-1}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta)\big)\mathscr{H}_1 = -\sqrt{2}\mathrm{i}\mu_\star\big(r - \mathrm{i}a\cos(\theta)\big)\,\mathscr{J}_2$$

$$r_+^{-1}\big(\Delta\,\partial_r + r - M - \mathrm{i}\omega\,(\Delta + 4Mr) - 2\mathrm{i}ak\big)\,\mathscr{J}_2 - \big(\partial_\theta + 2^{-1}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta)\big)\,\mathscr{J}_1$$
$$= \sqrt{2}\mathrm{i}\mu_\star\big(r + \mathrm{i}a\cos(\theta)\big)\mathscr{H}_2$$

$$r_+\big(\partial_r + \mathrm{i}\omega\big)\,\mathscr{J}_1 + \big(\partial_\theta + 2^{-1}\cot(\theta) - a\omega\sin(\theta) - k\csc(\theta)\big)\,\mathscr{J}_2 = -\sqrt{2}\mathrm{i}\mu_\star\big(r + \mathrm{i}a\cos(\theta)\big)\mathscr{H}_1\,.$$

This system is separable by means of the product ansatz

$$\begin{aligned}
\mathscr{H}_1(r,\theta) &= \mathscr{R}_+(r)\mathscr{T}_+(\theta)\\
\mathscr{H}_2(r,\theta) &= \mathscr{R}_-(r)\mathscr{T}_-(\theta)\\
\mathscr{J}_1(r,\theta) &= \mathscr{R}_-(r)\mathscr{T}_+(\theta)\\
\mathscr{J}_2(r,\theta) &= \mathscr{R}_+(r)\mathscr{T}_-(\theta)\,,
\end{aligned}$$

leading to a radial ODE system

$$\begin{aligned}
&\big(\Delta\,\partial_r + r - M - \mathrm{i}\omega\,(\Delta + 4Mr) - 2\mathrm{i}ak\big)\mathscr{R}_+ = r_+\big(\xi_{1/3} + \sqrt{2}\mathrm{i}\mu_\star r\big)\mathscr{R}_-\\
&r_+\big(\partial_r + \mathrm{i}\omega\big)\mathscr{R}_- = \big(\xi_{2/4} - \sqrt{2}\mathrm{i}\mu_\star r\big)\mathscr{R}_+
\end{aligned} \tag{11}$$

and to an angular ODE system

$$\begin{aligned}
&\big(\partial_\theta + 2^{-1}\cot(\theta) - a\omega\sin(\theta) - k\csc(\theta)\big)\mathscr{T}_- = -\big(\xi_{1/4} - \sqrt{2}\mu_\star a\cos(\theta)\big)\mathscr{T}_+\\
&\big(\partial_\theta + 2^{-1}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta)\big)\mathscr{T}_+ = \big(\xi_{2/3} + \sqrt{2}\mu_\star a\cos(\theta)\big)\mathscr{T}_-
\end{aligned} \tag{12}$$

with the constants of separation ξ_j, $j \in \{1,2,3,4\}$. From the radial system, it can be directly seen that the identifications $\xi_1 = \xi_3$ and $\xi_2 = \xi_4$ have to hold, whereas from the angular system, we obtain the identifications $\xi_1 = \xi_4$ and $\xi_2 = \xi_3$. Thus, defining $\xi := \xi_1 = \xi_2 = \xi_3 = \xi_4$, (11) and (12) reduce to

$$\begin{aligned}
&\big(\Delta\,\partial_r + r - M - \mathrm{i}\omega\,(\Delta + 4Mr) - 2\mathrm{i}ak\big)\mathscr{R}_+ = r_+\big(\xi + \sqrt{2}\mathrm{i}\mu_\star r\big)\mathscr{R}_-\\
&r_+\big(\partial_r + \mathrm{i}\omega\big)\mathscr{R}_- = \big(\xi - \sqrt{2}\mathrm{i}\mu_\star r\big)\mathscr{R}_+
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&\big(\partial_\theta + 2^{-1}\cot(\theta) - a\omega\sin(\theta) - k\csc(\theta)\big)\mathscr{T}_- = -\big(\xi - \sqrt{2}\mu_\star a\cos(\theta)\big)\mathscr{T}_+\\
&\big(\partial_\theta + 2^{-1}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta)\big)\mathscr{T}_+ = \big(\xi + \sqrt{2}\mu_\star a\cos(\theta)\big)\mathscr{T}_-\,,
\end{aligned} \tag{14}$$

respectively. The radial system (13) can be transformed into a more symmetric form by means of the functions $\widetilde{\mathscr{R}}_+ = \sqrt{|\Delta|}\,\mathscr{R}_+$ and $\widetilde{\mathscr{R}}_- = r_+\mathscr{R}_-$, yielding

$$\begin{aligned}
&\big(\Delta\,\partial_r - \mathrm{i}\omega\,(\Delta + 4Mr) - 2\mathrm{i}ak\big)\widetilde{\mathscr{R}}_+ = \sqrt{|\Delta|}\,\big(\xi + \sqrt{2}\mathrm{i}\mu_\star r\big)\widetilde{\mathscr{R}}_-\\
&\Delta\big(\partial_r + \mathrm{i}\omega\big)\widetilde{\mathscr{R}}_- = \operatorname{sign}(\Delta)\,\sqrt{|\Delta|}\,\big(\xi - \sqrt{2}\mathrm{i}\mu_\star r\big)\widetilde{\mathscr{R}}_+\,.
\end{aligned} \tag{15}$$

For the following study of the asymptotics and the decay of the radial solutions at infinity, the event horizon, and the Cauchy horizon, it is convenient to rewrite (15) in the matrix representation

$$\partial_r\widetilde{\mathscr{R}} = U(r)\widetilde{\mathscr{R}}\,, \tag{16}$$

where $\widetilde{\mathscr{R}} := \big(\widetilde{\mathscr{R}}_+, \widetilde{\mathscr{R}}_-\big)^T$ and

$$U(r) := \frac{1}{\Delta}\begin{pmatrix} \mathrm{i}\,\big(\omega(\Delta + 4Mr) + 2ak\big) & \sqrt{|\Delta|}\,\big(\xi + \sqrt{2}\mathrm{i}\mu_\star r\big)\\ \operatorname{sign}(\Delta)\,\sqrt{|\Delta|}\,\big(\xi - \sqrt{2}\mathrm{i}\mu_\star r\big) & -\mathrm{i}\omega\Delta \end{pmatrix}.$$

This representation also allows for a straightforward determination of the singular points of the radial system [9]. We find that U has singularities of rank $\mu = 0$ at $r = r_\pm$. Therefore, even though both the coordinate system and the tetrad frame are regular at the event and the Cauchy horizon, these are regular singular points of (15). Thus, the regions outside the event horizon $r_+ < r$, between the event and the Cauchy horizon $r_- < r < r_+$, and inside the Cauchy horizon $r < r_-$ have to be considered separately. Note that this is a particularity of the mode ansatz (10). Finally, the matrix representation of the angular system (14), with $\mathscr{T} := (\mathscr{T}_+, \mathscr{T}_-)^T$, is given by

$$\begin{pmatrix} \sqrt{2}\mu_\star a \cos(\theta) & -\partial_\theta - 2^{-1}\cot(\theta) + a\omega \sin(\theta) + k\csc(\theta) \\ \partial_\theta + 2^{-1}\cot(\theta) + a\omega\sin(\theta) + k\csc(\theta) & -\sqrt{2}\mu_\star a\cos(\theta) \end{pmatrix} \mathscr{T} = \xi \mathscr{T} \,. \tag{17}$$

B. Asymptotic Analysis of the Radial Solution at Infinity

Following the approach of [17], we derive the asymptotic solution of the radial system (16) for $r \to \infty$ and examine its decay. This solution is required (in addition to the asymptotics at the event and the Cauchy horizon, which are determined in the subsequent subsections), e.g., for a description of the scattering process of Dirac waves by the gravitational field of a Kerr black hole (cf. Section IV) and for the construction of a functional analytic integral representation of the Dirac propagator [18]. We begin by expressing (16) in terms of the Regge-Wheeler coordinate

$$\partial_{r_\star} \widetilde{\mathscr{R}} = T(r_\star) \widetilde{\mathscr{R}} \,, \tag{18}$$

where $T(r_\star) := \Delta/(r^2 + a^2)\, U(r_\star)$. Then, we represent this radial system by the diagonal matrix $S := D^{-1}TD = \operatorname{diag}(\lambda_1, \lambda_2)$, with $\lambda_{1/2}$ being the eigenvalues of T, and by the corresponding diagonalization matrix D

$$\partial_{r_\star}\left(D^{-1}\widetilde{\mathscr{R}}\right) = \left[S - D^{-1}\left(\partial_{r_\star} D\right)\right]\left(D^{-1}\widetilde{\mathscr{R}}\right).$$

Using the ansatz

$$\widetilde{\mathscr{R}}(r_\star) = D(r_\star) \begin{pmatrix} e^{\mathrm{i}\phi_+(r_\star)}\, \mathfrak{f}_1(r_\star) \\ e^{-\mathrm{i}\phi_-(r_\star)}\, \mathfrak{f}_2(r_\star) \end{pmatrix},$$

we obtain an ODE system for $\mathfrak{f} := (\mathfrak{f}_1, \mathfrak{f}_2)^T$

$$\partial_{r_\star}\mathfrak{f} = \left[S - W^{-1}D^{-1}\left(\partial_{r_\star} D\right) W - W^{-1}\partial_{r_\star} W\right]\mathfrak{f} \,,$$

where $W := \operatorname{diag}\left(e^{\mathrm{i}\phi_+}, e^{-\mathrm{i}\phi_-}\right)$. The functions $\phi_\pm$ are fixed by requiring that $S = W^{-1}\partial_{r_\star} W$, i.e.,

$$\partial_{r_\star}\phi_+ = -\mathrm{i}\lambda_1 \quad \text{and} \quad \partial_{r_\star}\phi_- = \mathrm{i}\lambda_2 \,, \tag{19}$$

yielding

$$\partial_{r_\star}\mathfrak{f} = -W^{-1}D^{-1}\left(\partial_{r_\star} D\right) W \,\mathfrak{f} \,. \tag{20}$$

Lemma III.1. *Every nontrivial solution $\widetilde{\mathscr{R}}$ of (18) is asymptotically as $r \to \infty$ of the form*

$$\widetilde{\mathscr{R}}(r_\star) = \widetilde{\mathscr{R}}_\infty(r_\star) + E_\infty(r_\star) = D_\infty \begin{pmatrix} e^{\mathrm{i}\phi_+(r_\star)}\, \mathfrak{f}_\infty^{(1)} \\ e^{-\mathrm{i}\phi_-(r_\star)}\, \mathfrak{f}_\infty^{(2)} \end{pmatrix} + E_\infty(r_\star) \tag{21}$$

with the asymptotic diagonalization matrix

$$D_\infty = \begin{cases} \begin{pmatrix} \cosh(\Omega) & \sinh(\Omega) \\ \sinh(\Omega) & \cosh(\Omega) \end{pmatrix} & \text{for} \;\; \omega^2 \geq 2\mu_\star^2 \\[2ex] \dfrac{1}{\sqrt{2}} \begin{pmatrix} \cosh(\Omega) + \mathrm{i}\sinh(\Omega) & \sinh(\Omega) + \mathrm{i}\cosh(\Omega) \\ \sinh(\Omega) + \mathrm{i}\cosh(\Omega) & \cosh(\Omega) + \mathrm{i}\sinh(\Omega) \end{pmatrix} & \text{for} \;\; \omega^2 < 2\mu_\star^2 \,, \end{cases} \tag{22}$$

where

$$\Omega := \begin{cases} \dfrac{1}{4}\ln\left(\dfrac{\omega-\sqrt{2}\mu_\star}{\omega+\sqrt{2}\mu_\star}\right) & \text{for } \omega^2 \geq 2\mu_\star^2 \\[2ex] \dfrac{1}{4}\ln\left(\dfrac{\sqrt{2}\mu_\star-\omega}{\sqrt{2}\mu_\star+\omega}\right) & \text{for } \omega^2 < 2\mu_\star^2\,, \end{cases} \tag{23}$$

the asymptotic phases

$$\phi_\pm(r_\star) \simeq \operatorname{sign}(\omega)\times \begin{cases} -\sqrt{\omega^2-2\mu_\star^2}\,r_\star + 2M\left(\pm\,\omega - \dfrac{\mu_\star^2}{\sqrt{\omega^2-2\mu_\star^2}}\right)\ln(r_\star) & \text{for } \omega^2 \geq 2\mu_\star^2 \\[2ex] \mathrm{i}\sqrt{2\mu_\star^2-\omega^2}\,r_\star + 2M\left(\pm\,\omega - \dfrac{\mathrm{i}\mu_\star^2}{\sqrt{2\mu_\star^2-\omega^2}}\right)\ln(r_\star) & \text{for } \omega^2 < 2\mu_\star^2\,, \end{cases} \tag{24}$$

and an error E_∞ with polynomial decay

$$\|E_\infty(r_\star)\| = \left\|\widetilde{\mathscr{R}}(r_\star) - \widetilde{\mathscr{R}}_\infty(r_\star)\right\| \leq \frac{a}{r_\star} \tag{25}$$

for a suitable constant $a \in \mathbb{R}_{>0}$. The asymptotics of the function $\mathfrak{f}$ for large values of r is given by

$$\mathfrak{f}_\infty := \left(\mathfrak{f}_\infty^{(1)}, \mathfrak{f}_\infty^{(2)}\right)^T = \text{const.} \neq \mathbf{0}$$

with an error $E_\mathfrak{f}$ that has polynomial decay

$$\|E_\mathfrak{f}(r_\star)\| = \|\mathfrak{f}(r_\star) - \mathfrak{f}_\infty\| \leq \frac{b}{r_\star}$$

for a suitable constant $b \in \mathbb{R}_{>0}$.

Proof. The matrix T defined below (18) converges for $r \to \infty$ to

$$T_\infty := \lim_{r_\star\to\infty} T = \mathrm{i}\begin{pmatrix} \omega & \sqrt{2}\mu_\star \\ -\sqrt{2}\mu_\star & -\omega \end{pmatrix}. \tag{26}$$

Further, it has a regular expansion in powers of $1/r_\star$, i.e., $T = T_\infty + \mathcal{O}(1/r_\star)$. As a consequence, both the diagonal matrix S and the diagonalization matrix D also have regular expansions in powers of $1/r_\star$. The eigenvalues of (26) read

$$\lambda_{1/2} \simeq \operatorname{sign}(\omega)\times \begin{cases} \mp\,\mathrm{i}\sqrt{\omega^2-2\mu_\star^2} \in \mathbb{C} & \text{for } \omega^2 \geq 2\mu_\star^2 \\ \mp\sqrt{2\mu_\star^2-\omega^2} \in \mathbb{R} & \text{for } \omega^2 < 2\mu_\star^2\,. \end{cases}$$

The associated matrix $D_\infty := \lim_{r_\star\to\infty} D$ is given by (22) with arguments (23). This can be easily verified by direct calculation. With the asymptotic first-order eigenvalues of T

$$\lambda_{1/2} \simeq \operatorname{sign}(\omega)\times \begin{cases} \mp\,\mathrm{i}\sqrt{\omega^2-2\mu_\star^2} + \dfrac{2\mathrm{i}M}{r_\star}\left(\omega \mp \dfrac{\mu_\star^2}{\sqrt{\omega^2-2\mu_\star^2}}\right) & \text{for } \omega^2 \geq 2\mu_\star^2 \\[2ex] \mp\sqrt{2\mu_\star^2-\omega^2} + \dfrac{2M}{r_\star}\left(\mathrm{i}\omega \pm \dfrac{\mu_\star^2}{\sqrt{2\mu_\star^2-\omega^2}}\right) & \text{for } \omega^2 < 2\mu_\star^2\,, \end{cases}$$

we can directly solve (19) by simple integration and obtain (24) for the asymptotic phases. Next, as the Hilbert-Schmidt norms of D^{-1} and $\partial_{r_\star} D$ are bounded from above by

$$\|D^{-1}\|_{\mathrm{HS}} \leq c \quad \text{and} \quad \|\partial_{r_\star} D\|_{\mathrm{HS}} \leq \frac{d}{r_\star^2}$$

for $r_\star$ sufficiently close to infinity, where both c and d denote positive constants, we estimate the $\mathbb{C}^2$-norm of (20) by

$$\|\partial_{r_\star}\mathfrak{f}\| \leq 2\,\|D^{-1}\|_{\mathrm{HS}} \cdot \|\partial_{r_\star} D\|_{\mathrm{HS}} \cdot \|\mathfrak{f}\| \leq \frac{2cd}{r_\star^2}\,\|\mathfrak{f}\| \tag{27}$$

with $\|W\|_{\mathrm{HS}} = \|W^{-1}\|_{\mathrm{HS}} = \sqrt{2}$. Employing the triangle and the Cauchy-Schwarz inequality, we derive the following inequality

$$\left|\partial_{r_\star}\|\mathfrak{f}\|\right| = \frac{|\partial_{r_\star}\langle\mathfrak{f},\mathfrak{f}\rangle|}{2\,\|\mathfrak{f}\|} = \frac{|\langle\mathfrak{f},\partial_{r_\star}\mathfrak{f}\rangle + \langle\partial_{r_\star}\mathfrak{f},\mathfrak{f}\rangle|}{2\,\|\mathfrak{f}\|} \leq \frac{|\langle\mathfrak{f},\partial_{r_\star}\mathfrak{f}\rangle| + |\langle\partial_{r_\star}\mathfrak{f},\mathfrak{f}\rangle|}{2\,\|\mathfrak{f}\|} = \frac{|\langle\mathfrak{f},\partial_{r_\star}\mathfrak{f}\rangle|}{\|\mathfrak{f}\|} \leq \frac{\|\mathfrak{f}\|\cdot\|\partial_{r_\star}\mathfrak{f}\|}{\|\mathfrak{f}\|} = \|\partial_{r_\star}\mathfrak{f}\|\,.$$

Consequently, using (27), we get

$$\left|\partial_{r_\star}\|\mathfrak{f}\|\right| \leq \frac{2cd}{r_\star^2}\,\|\mathfrak{f}\|\,. \tag{28}$$

Note that $\|\mathfrak{f}\| \neq 0$ because $\widetilde{\mathscr{R}}$ has to be nontrivial. Integrating (28) over the Regge-Wheeler coordinate from r_0 to $r_\star$ and applying the triangle inequality for integrals gives for all $0 < r_0 \leq r_\star$

$$\left|\int_{r_0}^{r_\star} \partial_{r'_\star} \ln\|\mathfrak{f}\|\,\mathrm{d}r'_\star\right| \leq \int_{r_0}^{r_\star} \left|\partial_{r'_\star}\ln\|\mathfrak{f}\|\right|\mathrm{d}r'_\star \leq 2cd\int_{r_0}^{r_\star}\frac{\mathrm{d}r'_\star}{r'^2_\star}$$

and, hence,

$$\left|\ln\|\mathfrak{f}\|\Big|_{r_0}^{r_\star}\right| \leq -\frac{2cd}{r'_\star}\Big|_{r_0}^{r_\star}.$$

Then, since $0 < 2cd/r'_\star\big|_{r_\star}^{r_0} < \infty$, there exists a constant $N > 0$ such that

$$\frac{1}{N} \leq \|\mathfrak{f}\| \leq N\,. \tag{29}$$

Substituting this into (27), we find for sufficiently large $r_\star$

$$\|\partial_{r_\star}\mathfrak{f}\| \leq \frac{b}{r_\star^2}\,, \tag{30}$$

where $b := 2cdN$. This implies that $\mathfrak{f}$ is integrable and, according to (29), has a finite, non-zero limit $\mathfrak{f}_\infty := \lim_{r_\star\to\infty}\mathfrak{f}(r_\star) \neq \mathbf{0}$. Finally, integrating (30) from $r_\star$ to ∞ and again using the triangle inequality for integrals, we obtain the error estimate

$$\|E_{\mathfrak{f}}\| = \|\mathfrak{f} - \mathfrak{f}_\infty\| = \left\|\int_{r_\star}^{\infty}\partial_{r'_\star}\mathfrak{f}\,\mathrm{d}r'_\star\right\| \leq \int_{r_\star}^{\infty}\|\partial_{r'_\star}\mathfrak{f}\|\,\mathrm{d}r'_\star \leq \frac{b}{r_\star}\,. \tag{31}$$

The $1/r_\star$-decay of the error E_∞ in (25) follows directly from (31). ■

C. Asymptotic Analysis of the Radial Solution at the Event Horizon

Using the solution ansatz

$$\widetilde{\mathscr{R}} = \begin{pmatrix} e^{2\mathrm{i}\left(\omega + k\Omega^{(+)}_{\mathrm{Kerr}}\right)r_\star}\,\mathfrak{g}_1(r_\star) \\ \mathfrak{g}_2(r_\star)\end{pmatrix},$$

with the angular velocity of the event horizon $\Omega^{(+)}_{\mathrm{Kerr}} := a/(2Mr_+)$, in (18) yields an ODE system for $\mathfrak{g} := (\mathfrak{g}_1, \mathfrak{g}_2)^T$

$$\partial_{r_\star}\mathfrak{g} = -\frac{\mathrm{i}}{r^2+a^2}\Bigg[2k\left(2M\Omega^{(+)}_{\mathrm{Kerr}}\,r - a\right)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \sqrt{\Delta} \\ \times \begin{pmatrix} \sqrt{\Delta}\left(\omega + 2k\Omega^{(+)}_{\mathrm{Kerr}}\right) & -e^{-2\mathrm{i}\left(\omega+k\Omega^{(+)}_{\mathrm{Kerr}}\right)r_\star}\left(\sqrt{2}\mu_\star r - \mathrm{i}\xi\right) \\ e^{2\mathrm{i}\left(\omega+k\Omega^{(+)}_{\mathrm{Kerr}}\right)r_\star}\left(\sqrt{2}\mu_\star r + \mathrm{i}\xi\right) & \sqrt{\Delta}\,\omega \end{pmatrix}\Bigg]\mathfrak{g}\,. \tag{32}$$

In the limit $r \searrow r_+$, the right-hand side of this system vanishes and, thus, we obtain the asymptotic solution $\mathfrak{g}_{r_+} := \lim_{r \searrow r_+} \mathfrak{g} = \text{const.}$

Lemma III.2. *Every nontrivial solution $\widetilde{\mathscr{R}}$ of (18) is asymptotically as $r \searrow r_+$ of the form*

$$\widetilde{\mathscr{R}}(r_\star) = \widetilde{\mathscr{R}}_{r_+}(r_\star) + E_{r_+}(r_\star) = \begin{pmatrix} e^{2\mathrm{i}\left(\omega + k\Omega^{(+)}_{\mathrm{Kerr}}\right) r_\star}\, \mathfrak{g}^{(1)}_{r_+} \\ \mathfrak{g}^{(2)}_{r_+} \end{pmatrix} + E_{r_+}(r_\star) \tag{33}$$

with

$$\mathfrak{g}_{r_+} := \left(\mathfrak{g}^{(1)}_{r_+}, \mathfrak{g}^{(2)}_{r_+}\right)^T = \text{const.} \neq \mathbf{0}$$

and an error E_{r_+} with exponential decay

$$\|E_{r_+}(r_\star)\| \leq p\, e^{q r_\star}$$

for r sufficiently close to r_+ and suitable constants $p, q \in \mathbb{R}_{>0}$.

Proof. For $r \searrow r_+$, as $r \simeq r_+ + e^{2qr_\star}$ with $q := (r_+ - r_-)/(4Mr_+) \in \mathbb{R}_{>0}$, the right-hand side of (32) is of the order $\mathcal{O}\big(e^{qr_\star}\big)$. Hence, there exists a constant $p' \in \mathbb{R}_{>0}$ such that for $r_\star$ sufficiently close to $-\infty$

$$\|\partial_{r_\star} \mathfrak{g}\| \leq p' e^{qr_\star} \|\mathfrak{g}\|\,. \tag{34}$$

Similar to the proof of the previous subsection, we find for all $r_\star \leq r_0$

$$\left| \ln \|\mathfrak{g}\| \Big|_{r_\star}^{r_0} \right| \leq \frac{p'}{q}\, e^{qr'_\star} \Big|_{r_\star}^{r_0}\,,$$

and since $e^{qr'_\star}\big|_{r_\star}^{r_0}$ is positive, there is a constant $N' > 0$ such that the $\mathbb{C}^2$-norm of $\mathfrak{g}$ is bounded

$$\frac{1}{N'} \leq \|\mathfrak{g}\| \leq N'\,. \tag{35}$$

Substituting (35) into (34) yields

$$\|\partial_{r_\star} \mathfrak{g}\| \leq p\, e^{qr_\star}\,, \tag{36}$$

where $p := p'N'$. All this implies that $\mathfrak{g}$ is integrable and has a finite, non-zero limit for $r_\star \to -\infty$. Integrating (36) from $-\infty$ to r_0 and applying the triangle inequality for integrals, we obtain

$$\|E_{\mathfrak{g}}\| = \|\mathfrak{g} - \mathfrak{g}_{r_+}\| = \left\| \int_{-\infty}^{r_\star} \partial_{r'_\star} \mathfrak{g}\, \mathrm{d}r'_\star \right\| \leq \int_{-\infty}^{r_\star} \|\partial_{r'_\star} \mathfrak{g}\|\, \mathrm{d}r'_\star \leq p\, e^{qr_\star}\,, \tag{37}$$

which proves the exponential decay of the error $E_{\mathfrak{g}}$. The exponential decay of the error E_{r_+} follows directly from (37). ■

D. Asymptotic Analysis of the Radial Solution at the Cauchy Horizon

We begin with the solution ansatz

$$\widetilde{\mathscr{R}} = \begin{pmatrix} e^{2\mathrm{i}\left(\omega + k\Omega^{(-)}_{\mathrm{Kerr}}\right) r_\star}\, \mathfrak{h}_1(r_\star) \\ \mathfrak{h}_2(r_\star) \end{pmatrix},$$

where $\Omega^{(-)}_{\mathrm{Kerr}} := a/(2Mr_-)$ is the angular velocity of the Cauchy horizon, and employ it in (18). This leads to an ODE system for $\mathfrak{h} := (\mathfrak{h}_1, \mathfrak{h}_2)^T$

$$\partial_{r_\star} \mathfrak{h} = \frac{\mathrm{i}}{r^2 + a^2} \Bigg[-2k\left(2M\Omega^{(-)}_{\mathrm{Kerr}} r - a\right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sqrt{|\Delta|}$$

$$\times \begin{pmatrix} \sqrt{|\Delta|}\left(\omega + 2k\Omega^{(-)}_{\mathrm{Kerr}}\right) & e^{-2\mathrm{i}\left(\omega + k\Omega^{(-)}_{\mathrm{Kerr}}\right) r_\star}\left(\sqrt{2}\mu_\star r - \mathrm{i}\xi\right) \\ e^{2\mathrm{i}\left(\omega + k\Omega^{(-)}_{\mathrm{Kerr}}\right) r_\star}\left(\sqrt{2}\mu_\star r + \mathrm{i}\xi\right) & \sqrt{|\Delta|}\,\omega \end{pmatrix} \Bigg] \mathfrak{h}\,.$$

In the limit $r \searrow r_-$, the square bracket on the right-hand side vanishes, resulting in the asymptotic solution $\mathfrak{h}_{r_-} := \lim_{r \searrow r_-} \mathfrak{h} = \text{const.}$

Lemma III.3. *Every nontrivial solution $\widetilde{\mathscr{R}}$ of (18) is asymptotically as $r \searrow r_-$ of the form*

$$\widetilde{\mathscr{R}}(r_\star) = \widetilde{\mathscr{R}}_{r_-}(r_\star) + E_{r_-}(r_\star) = \begin{pmatrix} e^{2\mathrm{i}\left(\omega + k\Omega_{\text{Kerr}}^{(-)}\right)r_\star}\,\mathfrak{h}_{r_-}^{(1)} \\ \mathfrak{h}_{r_-}^{(2)} \end{pmatrix} + E_{r_-}(r_\star)$$

with

$$\mathfrak{h}_{r_-} := \left(\mathfrak{h}_{r_-}^{(1)}, \mathfrak{h}_{r_-}^{(2)}\right)^T = \text{const.} \neq \mathbf{0}$$

and an error E_{r_-} with exponential decay

$$\|E_{r_-}(r_\star)\| \leq u\, e^{-v r_\star}$$

for r sufficiently close to r_- and suitable constants $u, v \in \mathbb{R}_{>0}$.

Proof. The proof of this lemma is similar to the proof of Lemma III.2. ■

E. Eigenfunctions and Eigenvalues of the Angular System

The angular system (17) in its decoupled second-order form is known as the massive Chandrasekhar-Page equation [8]. In the limit $a \searrow 0$, the solutions of this equation reduce to the spin-weighted spherical harmonics for the spin-1/2 case [21]. For non-zero angular momenta, the solutions are usually referred to as the spin-1/2 spheroidal harmonics. For a good introduction and a compilation of some properties of these functions, we refer the reader to the recent paper [10]. In the present work, it is only of importance that the matrix-valued differential operator on the left-hand side of (17) has a spectral decomposition with discrete, non-degenerate eigenvalues and smooth eigenfunctions, which was proven in [15, 17]. In the following, we state the result.

Proposition III.4. *For any $\omega \in \mathbb{R}$ and $k \in \mathbb{Z} + 1/2$, the differential operator in (17) has a complete set of orthonormal eigenfunctions $(\mathscr{T}_n)_{n\in\mathbb{Z}}$ in $L^2\big((0,\pi), \sin(\theta)\,\mathrm{d}\theta\big)^2$. The corresponding eigenvalues ξ_n are real-valued and non-degenerate and can, thus, be ordered as $\xi_n < \xi_{n+1}$. Moreover, the eigenfunctions are bounded and smooth away from the poles. Both the eigenfunctions and the eigenvalues depend smoothly on ω.*

IV. SCATTERING OF MASSIVE DIRAC WAVES BY THE GRAVITATIONAL FIELD OF A KERR BLACK HOLE

In this section, we study the scattering of massive Dirac waves by the gravitational field of a Kerr black hole from the point of view of an observer described by horizon-penetrating advanced Eddington-Finkelstein-type coordinates. To this end, we consider Dirac waves that emerge from space-like infinity and propagate toward the black hole's event horizon. We compute the net current at infinity as well as at the horizon, and derive a conservation law for the reflexion and transmission coefficients. Starting with the asymptotic radial solution at infinity (21) and at the event horizon (33), we impose boundary conditions specifying an incident wave of unit amplitude at infinity, a reflected wave of amplitude $A_{\omega,\mu_\star}$ at infinity, and a transmitted wave of amplitude $B_{\omega,\mu_\star}$ at the event horizon. The asymptotic ingoing and outgoing radial solutions adapted to these boundary conditions read

$$\widetilde{\mathscr{R}}_{\text{Scat.}}(r \to \infty) \simeq \begin{pmatrix} A_{\omega,\mu_\star}\, e^{\mathrm{i}\phi_+(r_\star)} \\ e^{-\mathrm{i}\phi_-(r_\star)} \end{pmatrix} \tag{38}$$

and

$$\widetilde{\mathscr{R}}_{\text{Scat.}}(r \searrow r_+) \simeq \begin{pmatrix} 0 \\ B_{\omega,\mu_\star} \end{pmatrix}. \tag{39}$$

We point out that the boundary conditions are chosen in conformity with the physical requirement that no waves can emerge from the event horizon. Besides, only the asymptotic solution branch of (21) with $\omega^2 \geq 2\mu_\star^2$

is considered because free particles at infinity must have energies that exceed – or at least equal – their rest energies. Next, assuming the normalization condition $\|\mathscr{T}_+(\theta)\|^2 + \|\mathscr{T}_-(\theta)\|^2 = 1$ for the angular eigenfunctions (cf. Proposition III.4), the radial Dirac current (for definitions and notations see Appendix B) yields

$$J^r = \sqrt{2}\,\sigma^r{}_{A\dot{B}}\left(P^A\,\overline{P}^{\dot{B}} + Q^A\,\overline{Q}^{\dot{B}}\right) = \frac{1}{r_+\Sigma}\left(\mathrm{sign}(\Delta)\,\|\widetilde{\mathscr{R}}_+\|^2 - \|\widetilde{\mathscr{R}}_-\|^2\right)$$

with the radial Infeld-van der Waerden symbol

$$\sigma^r{}_{A\dot{B}} = \begin{pmatrix} l^r & m^r \\ \overline{m}^r & n^r \end{pmatrix}_{A\dot{B}} = \frac{1}{\sqrt{2\,\Sigma}\,r_+}\begin{pmatrix} \Delta & 0 \\ 0 & -r_+^2 \end{pmatrix}_{A\dot{B}}.$$

As this current has jump discontinuities at the event and the Cauchy horizon (see the final paragraph of Section III A), the regions separated by the horizons have to be evaluated independently. For the scattering problem at hand, however, it suffices to consider the exterior region of the black hole $r_+ < r$, where the radial Dirac current reads

$$J^r = \frac{1}{r_+\Sigma}\left(\|\widetilde{\mathscr{R}}_+\|^2 - \|\widetilde{\mathscr{R}}_-\|^2\right). \tag{40}$$

From the radial system (15) and the corresponding conjugated complex system, we obtain the relation

$$\|\widetilde{\mathscr{R}}_-\|^2 - \|\widetilde{\mathscr{R}}_+\|^2 = \mathrm{const.}$$

by simple algebraic manipulations. Substituting this into (40), we derive the conserved net current of particles

$$\partial_t N = -\int_0^{2\pi}\int_0^{\pi} J^r\sqrt{|\det(\boldsymbol{g})|}\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{4\pi}{r_+}\left(\|\widetilde{\mathscr{R}}_-\|^2 - \|\widetilde{\mathscr{R}}_+\|^2\right) = \mathrm{const.}\,,$$

where N is the total number of particles and $\sqrt{|\det(\boldsymbol{g})|} = \Sigma\sin(\theta)$. Defining the reflexion and transmission coefficients

$$R_{\omega,\mu_\star} := |A_{\omega,\mu_\star}|^2 \quad \text{and} \quad T_{\omega,\mu_\star} := |B_{\omega,\mu_\star}|^2$$

and using (38) and (39), the net current at infinity and at the event horizon becomes

$$\partial_t N_{|r\to\infty} = \frac{4\pi}{r_+}\left(1 - R_{\omega,\mu_\star}\right) \tag{41}$$

and

$$\partial_t N_{|r\searrow r_+} = \frac{4\pi}{r_+}\,T_{\omega,\mu_\star}\,, \tag{42}$$

respectively. It follows from the latter equation that the net current across the event horizon is always positive. Furthermore, with the constancy of the net current and (41) and (42), we deduce

$$R_{\omega,\mu_\star} + T_{\omega,\mu_\star} = 1\,,$$

which proves that superradiance cannot occur because the reflexion coefficient is always less than unity. These results are in agreement with those found in the accordant analysis of this scattering problem employing Boyer-Lindquist coordinates (see [8] and references therein).

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Appendix A: The Newman-Penrose Formalism

Let $(\mathfrak{M}, \boldsymbol{g})$ be a Lorentzian 4-manifold endowed with the unique, torsion-free Levi-Civita connection $\boldsymbol{\omega}$ and dual basis $(\boldsymbol{e}_\mu)$ and $(\boldsymbol{e}^\mu)$, $\mu \in \{0,1,2,3\}$, on sections of the tangent and cotangent bundles $T\mathfrak{M}$ and $T^\star\mathfrak{M}$, respectively. Further, let $F\mathfrak{M}$ and $F^\star\mathfrak{M}$ be local (orthonormal or null) frame bundles on $\mathfrak{M}$. The dual basis on sections of $F\mathfrak{M}$ and $F^\star\mathfrak{M}$ each consist of four vectors denoted by $(\boldsymbol{e}_{(a)})$ and $(\boldsymbol{e}^{(a)})$, $a \in \{0,1,2,3\}$. In terms of the original basis vectors, these can be written as $\boldsymbol{e}_{(a)} = e^\mu{}_{(a)} \boldsymbol{e}_\mu$ and $\boldsymbol{e}^{(a)} = e_\mu{}^{(a)} \boldsymbol{e}^\mu$, where $e^\mu{}_{(a)}$ is a linear map from $T\mathfrak{M}$ to $F\mathfrak{M}$, namely a (4×4)-matrix, and $e_\mu{}^{(a)}$ is its inverse. In the Newman-Penrose formalism [25], the local basis vectors are given by two real-valued null vectors, $\boldsymbol{l} = \boldsymbol{e}_{(0)} = \boldsymbol{e}^{(1)}$ and $\boldsymbol{n} = \boldsymbol{e}_{(1)} = \boldsymbol{e}^{(0)}$, as well as by a conjugate-complex pair of null vectors, $\boldsymbol{m} = \boldsymbol{e}_{(2)} = -\boldsymbol{e}^{(3)}$ and $\overline{\boldsymbol{m}} = \boldsymbol{e}_{(3)} = -\boldsymbol{e}^{(2)}$. These have to satisfy the null conditions

$$\boldsymbol{l}\cdot\boldsymbol{l} = \boldsymbol{n}\cdot\boldsymbol{n} = \boldsymbol{m}\cdot\boldsymbol{m} = \overline{\boldsymbol{m}}\cdot\overline{\boldsymbol{m}} = 0\,, \tag{A1}$$

the orthogonality conditions

$$\boldsymbol{l}\cdot\boldsymbol{m} = \boldsymbol{l}\cdot\overline{\boldsymbol{m}} = \boldsymbol{n}\cdot\boldsymbol{m} = \boldsymbol{n}\cdot\overline{\boldsymbol{m}} = 0\,, \tag{A2}$$

and the cross-normalization conditions (which depend on the signature convention)

$$\boldsymbol{l}\cdot\boldsymbol{n} = -\boldsymbol{m}\cdot\overline{\boldsymbol{m}} = 1\,. \tag{A3}$$

The local metric in the Newman-Penrose formalism is non-degenerate and constant. It reads

$$\boldsymbol{\eta} = g_{\mu\nu} e^\mu{}_{(a)} e^\nu{}_{(b)} \boldsymbol{e}^{(a)} \otimes \boldsymbol{e}^{(b)} = \boldsymbol{l}\otimes\boldsymbol{n} + \boldsymbol{n}\otimes\boldsymbol{l} - \boldsymbol{m}\otimes\overline{\boldsymbol{m}} - \overline{\boldsymbol{m}}\otimes\boldsymbol{m}\,.$$

Since the Levi-Civita connection is torsion-free, the first Maurer-Cartan equation of structure simplifies to

$$\mathrm{d}\boldsymbol{e}^{(a)} = \gamma^{(a)}{}_{(b)(c)} \boldsymbol{e}^{(b)} \wedge \boldsymbol{e}^{(c)}\,, \tag{A4}$$

where the Ricci rotation coefficients $\gamma^{(a)}{}_{(b)(c)}$ are related to the connection via

$$\gamma^{(a)}{}_{(b)(c)} \boldsymbol{e}^{(c)} = e_\mu{}^{(a)} \mathrm{d}e^\mu{}_{(b)} + e_\mu{}^{(a)} e^\nu{}_{(b)}\, \omega^\mu{}_\nu\,.$$

In the Newman-Penrose formalism, (A4) becomes

$$\begin{aligned}
\mathrm{d}\boldsymbol{l} &= 2\Re(\epsilon)\,\boldsymbol{n}\wedge\boldsymbol{l} - 2\boldsymbol{n}\wedge\Re(\kappa\,\overline{\boldsymbol{m}}) - 2\boldsymbol{l}\wedge\Re\big([\tau-\overline{\alpha}-\beta]\,\overline{\boldsymbol{m}}\big) + 2\mathrm{i}\,\Im(\varrho)\,\boldsymbol{m}\wedge\overline{\boldsymbol{m}} \\[1em]
\mathrm{d}\boldsymbol{n} &= 2\Re(\gamma)\,\boldsymbol{n}\wedge\boldsymbol{l} - 2\boldsymbol{n}\wedge\Re\big([\overline{\alpha}+\beta-\overline{\pi}]\,\overline{\boldsymbol{m}}\big) + 2\boldsymbol{l}\wedge\Re(\overline{\nu}\,\overline{\boldsymbol{m}}) + 2\mathrm{i}\,\Im(\mu)\,\boldsymbol{m}\wedge\overline{\boldsymbol{m}} \\[1em]
\mathrm{d}\boldsymbol{m} &= \overline{\mathrm{d}\overline{\boldsymbol{m}}} = (\overline{\pi}+\tau)\,\boldsymbol{n}\wedge\boldsymbol{l} + \big(2\mathrm{i}\,\Im(\epsilon)-\varrho\big)\,\boldsymbol{n}\wedge\boldsymbol{m} - \sigma\,\boldsymbol{n}\wedge\overline{\boldsymbol{m}} + \big(\overline{\mu}+2\mathrm{i}\,\Im(\gamma)\big)\,\boldsymbol{l}\wedge\boldsymbol{m} \\
&\qquad + \overline{\lambda}\,\boldsymbol{l}\wedge\overline{\boldsymbol{m}} - (\overline{\alpha}-\beta)\,\boldsymbol{m}\wedge\overline{\boldsymbol{m}}
\end{aligned} \tag{A5}$$

with the so-called spin coefficients

$$\begin{aligned}
&\kappa = \gamma_{(2)(0)(0)} && \varrho = \gamma_{(2)(0)(3)} && \epsilon = \tfrac{1}{2}\big(\gamma_{(1)(0)(0)} + \gamma_{(2)(3)(0)}\big) \\
&\sigma = \gamma_{(2)(0)(2)} && \mu = \gamma_{(1)(3)(2)} && \gamma = \tfrac{1}{2}\big(\gamma_{(1)(0)(1)} + \gamma_{(2)(3)(1)}\big) \\
&\lambda = \gamma_{(1)(3)(3)} && \tau = \gamma_{(2)(0)(1)} && \alpha = \tfrac{1}{2}\big(\gamma_{(1)(0)(3)} + \gamma_{(2)(3)(3)}\big) \\
&\nu = \gamma_{(1)(3)(1)} && \pi = \gamma_{(1)(3)(0)} && \beta = \tfrac{1}{2}\big(\gamma_{(1)(0)(2)} + \gamma_{(2)(3)(2)}\big)
\end{aligned} \tag{A6}$$

and the real and imaginary parts $\Re(\cdot)$ and $\Im(\cdot)$, respectively. The transformations employed in this study are elements of the 2-parameter subgroup of local Lorentz transformations known as class III or spin-boost Lorentz transformations. These renormalize the real-valued Newman-Penrose vectors $\boldsymbol{l}$ and $\boldsymbol{n}$, but leave their directions unchanged, and rotate the conjugate-complex pair $\boldsymbol{m}$ and $\overline{\boldsymbol{m}}$ by an angle ψ in the $(\boldsymbol{m},\overline{\boldsymbol{m}})$-plane [8]

$$\boldsymbol{l}\mapsto\boldsymbol{l}' = \varsigma\,\boldsymbol{l}\,,\quad \boldsymbol{n}\mapsto\boldsymbol{n}' = \varsigma^{-1}\,\boldsymbol{n}\,,\quad \boldsymbol{m}\mapsto\boldsymbol{m}' = e^{\mathrm{i}\psi}\boldsymbol{m}\,,\quad \overline{\boldsymbol{m}}\mapsto\overline{\boldsymbol{m}}' = e^{-\mathrm{i}\psi}\,\overline{\boldsymbol{m}}\,, \tag{A7}$$

where $\varsigma \in \mathbb{R}\backslash\{0\}$ and $\psi \in \mathbb{R}$ are functions depending on the spacetime coordinates (x^μ). There are various aspects of the Newman-Penrose formalism that are not discussed in this appendix such as the different classes of local Lorentz transformations or the Weyl scalars and their algebraic classification. They can be found elsewhere in the literature. The interested reader may be referred to [26, 28].

Appendix B: The General Relativistic Dirac Equation

The general relativistic, massive Dirac equation without an external potential is given by [19, 32]

$$\big(\gamma^\mu \nabla_\mu + \mathrm{i}m\big)\Psi(x^\mu) = 0\,,$$

where the γ^μ are the general relativistic Dirac matrices, which satisfy the anticommutator relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathrm{id}_{\mathbb{C}^4}$, $\Psi(x^\mu)$ is a Dirac 4-spinor on sections $S_x\mathfrak{M} \simeq \mathbb{C}^4$ of the spin bundle $S\mathfrak{M} = \mathfrak{M} \times \mathbb{C}^4$ on $\mathfrak{M}$, $\boldsymbol{\nabla}$ denotes the metric connection on $S\mathfrak{M}$, and m is the fermion rest mass. Using the chiral 2-spinor representation of the Dirac 4-spinors and matrices [20]

$$\Psi = \begin{pmatrix} P^A \\ \overline{Q}_{\dot{B}} \end{pmatrix} \quad \text{and} \quad \gamma^\mu = \sqrt{2}\begin{pmatrix} 0 & \sigma^{\mu A\dot{B}} \\ \sigma^\mu{}_{A\dot{B}} & 0 \end{pmatrix}\,, \qquad A \in \{1, 2\}\,, \qquad \dot{B} \in \{\dot{1}, \dot{2}\}\,,$$

with the 2-component spinors P^A and $\overline{Q}_{\dot{B}}$ and the Hermitian (2×2)-Infeld-van der Waerden symbols $\sigma^\mu{}_{A\dot{B}}$, we obtain the following 2-spinor form of the Dirac equation

$$\begin{aligned} \nabla_{A\dot{B}} P^A + \mathrm{i}\mu_\star \overline{Q}_{\dot{B}} &= 0 \\ \nabla_{A\dot{B}} Q^A + \mathrm{i}\mu_\star \overline{P}_{\dot{B}} &= 0\,, \end{aligned} \tag{B1}$$

where $\mu_\star := m/\sqrt{2}$ and $\nabla_{A\dot{B}} = \sigma^\mu{}_{A\dot{B}} \nabla_\mu$. Note that dotted indices are subjected to conjugated complex transformations. Next, let $\zeta_{(k)}$ and $\zeta^{(k)}$, $k \in \{1, 2\}$, be (dual) local Newman-Penrose bases for the Dirac 2-spinors. The associated local spinor components can be expressed by $\mathscr{Y}^{(k)} = \zeta^{(k)}{}_A \mathscr{Y}^A$ and $\mathscr{Y}_{(k)} = \zeta_{(k)}{}^A \mathscr{Y}_A$ with the original 2-spinor components $\mathscr{Y}^A, \mathscr{Y}_A \in \mathbb{C}^2$ as well as the (2×2)-matrix $\zeta^{(k)}{}_A$ and its inverse $\zeta_{(k)}{}^A$. In this representation, the metric connection reads

$$\nabla_{(k)(\dot{l})} \mathscr{Y}^{(m)} = \zeta^A{}_{(k)}\, \overline{\zeta}^{\dot{B}}{}_{(\dot{l})}\, \zeta^{(m)}{}_C \nabla_{A\dot{B}}\, \mathscr{Y}^C = \partial_{(k)(\dot{l})} \mathscr{Y}^{(m)} + \Gamma^{(m)}{}_{(n)(k)(\dot{l})} \mathscr{Y}^{(n)}\,, \tag{B2}$$

where $\partial_{(k)(\dot{l})} = \sigma^\mu{}_{(k)(\dot{l})} \partial_\mu$ and

$$\Gamma^{(m)}{}_{(n)(k)(\dot{l})} = \Gamma^{(m)(\dot{o})}{}_{(n)(\dot{o})(k)(\dot{l})} = \sqrt{2}\, \epsilon^{(m)(q)}\, \epsilon^{(\dot{o})(\dot{p})}\, \sigma^\mu{}_{(q)(\dot{p})}\, \sigma^\nu{}_{(n)(\dot{o})}\, \sigma^\lambda{}_{(k)(\dot{l})}\, e_\mu{}^{(a)}\, e_\nu{}^{(b)}\, e_\lambda{}^{(c)}\, \gamma_{(a)(b)(c)}\,. \tag{B3}$$

We point out that the 2-dimensional Levi-Civita symbol $\boldsymbol{\epsilon}$ acts as skew metric on $\mathbb{C}^2$. Furthermore, the Infeld-van der Waerden symbols yield

$$\sigma^\mu{}_{(k)(\dot{l})} = \begin{pmatrix} l^\mu & m^\mu \\ \overline{m}^\mu & n^\mu \end{pmatrix}. \tag{B4}$$

By means of (B2)-(B4), (A6), and the definitions $\mathscr{F}_1 := P^{(1)}, \mathscr{F}_2 := P^{(2)}, \mathscr{G}_1 := \overline{Q}^{(\dot{2})}$, as well as $\mathscr{G}_2 := -\overline{Q}^{(\dot{1})}$, the general relativistic Dirac equation (B1) in the Newman-Penrose formalism becomes

$$\begin{aligned} (l^\mu \partial_\mu + \varepsilon - \varrho)\mathscr{F}_1 + (\overline{m}^\mu \partial_\mu + \pi - \alpha)\mathscr{F}_2 &= \mathrm{i}\mu_\star \mathscr{G}_1 \\ (n^\mu \partial_\mu + \mu - \gamma)\mathscr{F}_2 + (m^\mu \partial_\mu + \beta - \tau)\mathscr{F}_1 &= \mathrm{i}\mu_\star \mathscr{G}_2 \\ (l^\mu \partial_\mu + \overline{\varepsilon} - \overline{\varrho})\mathscr{G}_2 - (m^\mu \partial_\mu + \overline{\pi} - \overline{\alpha})\mathscr{G}_1 &= \mathrm{i}\mu_\star \mathscr{F}_2 \\ (n^\mu \partial_\mu + \overline{\mu} - \overline{\gamma})\mathscr{G}_1 - (\overline{m}^\mu \partial_\mu + \overline{\beta} - \overline{\tau})\mathscr{G}_2 &= \mathrm{i}\mu_\star \mathscr{F}_1\,. \end{aligned} \tag{B5}$$

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